JOURNAL OF DIFFERENTIAL EQUATIONS 70, 153–166 (1987)

Nonorientable Recurrence of Flows and Interval Exchange Transformations*

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Received January 24, 1986; revised August 26, 1986

INTRODUCTION

In [1], Peixoto proved that on an orientable surface, the Morse–Smale vector fields (see [2, p. 118]) are dense in the space of C^r vector fields, $r = 1, 2, \ldots$. Moreover, they are those which are structurally stable. However, the method he used to prove this result does not apply to a nonorientable surface. For special reasons the same fact holds for nonorientable surfaces of genus $\leqslant 3$ (see [3, 4]). Whether the Morse–Smale vector fields are dense on nonorientable surfaces of genus $\geqslant 4$ is still an open question. In fact, Gutierrez [5] showed that M^2 has at least one C^∞ vector field with nonorientable (nontrivial) recurrent trajectories: a trajectory γ such that if $p \in \gamma$ then $\gamma - \{p\}$ has two connected components, γ^+ and γ^-, and, if $p \in S$, S being a segment transverse to the flow, there exist connected components $ab \subset \gamma^+ - S$ and $cd \subset \gamma^- - S$ such that $ab \cup S$ and $cd \cup S$ contain a one-sided simple closed curve. The existence of such recurrent trajectories are the main obstacle in tackling the problem of density of the Morse–Smale vector fields.

A vector field on a surface of genus n, gives rise—through "cut and paste"—to a vector field on surfaces of greater genus, this is not necessarily the case if we consider a surface of smaller genus. The aim of this paper is to prove (Theorem 2) the existence of nondenumerable many C^∞ vector fields on any nonorientable surface of genus $n \geqslant 4$ which have nonorientable dense trajectories. These examples are such that the minimum genus of a surface where they can be defined is n. The richness of these examples shows that an extension of Peixoto's theorem for nonorientable surface of genus $\geqslant 4$ does not envolve just vector fields on a torus with two cross-caps (genus 4), but other surfaces as well. On the other hand, for the class of C^∞

* Supported in part by CNPq Grant 30.1456/80.

0022-0396/87 $3.00

vector fields on M^2 which correspond to measure-preserving Poincaré maps on a transverse circle, we prove (Theorem 3) that there are no non-trivial recurrences for most of these fields. Roughly speaking our result says that in this class the closure of the set of fields presenting nonorientable recurrent trajectories is a vanishing set. It follows that much vector fields can be C^∞ approximated by Morse–Smale fields.

The main point of the example given in [5] consists of finding a minimal (nonperiodic) interval exchange transformation (see Sect. 1 or Keane [6]) not preserving orientation in all intervals. Here we introduce a method of studying such transformations when they are minimal. Note that in the orientation preserving case much is known about these transformations; in particular, almost all of them are uniquely ergodic (see [6], Cornfield *et al.* [7], Masur [8], and Veech [9]). In the nonpreserving case, the opposite happens: almost all transformations are nonergodic (see the author [10]).

Let T be a minimal (dense orbit) interval exchange transformation defined on the interval $[0, 1)$.

These maps are closely related to certain vector fields on surfaces as follows. Let M^2 be a compact two-manifold and ϕ a continuous flow on M^2 with a dense positive semitrajectory and which has only finitely many singularities. Let C be a circle properly chosen transverse to ϕ and $\tilde{T}: C \to C$ the Poincaré map induced by ϕ on C. Gutierrez [11] showed that $\tilde{T}$ is conjugate to a nonperiodic interval exchange transformation T; that is, there exists a homeomorphism $h: C \to [0, 1)$ such that

$$\tilde{T} = h^{-1} \cdot T \cdot h.$$

Moreover, given such a map T, there exists a continuous flow ϕ on a surface M^2 and a circle C transverse to ϕ such that the Poincaré map induced by ϕ on C is conjugate to T.

In Section 1, we present a constructive method to derive examples of nonorientable interval exchange transformations like the one obtained in [5]. In Section 2, we prove Theorem 2, and Theorem 3 is proved in Section 3.

1. Interval Exchange Transformations

Let $I = [0, 1)$ and $n \geq 2$ be an integer. By a probability vector $\alpha = (\alpha_1, ..., \alpha_n)$ we mean $\alpha_i > 0$, for all i, and $\alpha_1 + \cdots + \alpha_n = 1$, we set $\beta_0 = 0$, $\beta_i = \sum_{j=1}^{i} \alpha_j$ and $I_i = [\beta_{i-1}, \beta_i)$. Let τ be a permutation on $\{1, 2, ..., n\}$. Then

$$\alpha^\tau = (\alpha_{\tau^{-1}(1)}, \alpha_{\tau^{-1}(2)}, ..., \alpha_{\tau^{-1}(n)})$$

is also a probability vector, and we can take the corresponding β_i^τ and I_i^τ, $1 \leqslant i \leqslant n$. Let $F \subset \{1, ..., n\}$ be the flip set. We now define $T = T_{(\tau,F,L)}: I \to I$, for each $x \in I_i$ and each $1 \leqslant i \leqslant n$, by

(i) if $i \notin F$ T preserves corientation on I_i

$$Tx = x - \beta_{i-1}\beta^\tau_{\tau(i)-1},$$

(ii) if $i \in F$ T reserves orientation on I_i

$$Tx = \beta_i - x + \beta^\tau_{\tau(i)-1}.$$

In the latter, T is not defined at the endpoint β_{i-1}.

We call T the interval exchange transformation. If F is non-empty, we say that T is an interval exchange transformation with flips.

Our first goal is to define an interval exchange transformation with flips T such that, for a suitable chosen subinterval the forward Poincaré map, T_1, induced by T behaves like T. Roughly speaking, T_1 is equal to T contracted to that subinterval. Therefore, T_1 also induces in a suitable interval a forward Poincaré map which is a contracted copy of T, and so on.

Let $L = (L_1, L_2, L_3, L_4, L_5)$ be a probability vector with positive components and $e = L_4 + L_5$. We denote by T the interval exchange transformation on the set $D = (0, 1) - \{L_1, L_1 + L_2, 1 - e, 1 - L_5\}$ given by

$$Tx = \begin{cases} e + x & \text{if } 0 < x < L_1 \\ 2L_1 + L_2 + e - x & \text{if } L_1 < x < L_1 + L_2 \\ e + x & \text{if } L_1 + L_2 < x < 1 - e \\ e + x - 1 & \text{if } 1 - e < x < 1 - L_5 \\ 2 + L_4 - e - x & \text{if } 1 - L_5 < x < 1. \end{cases} \tag{1.1}$$

T preserves orientation in $(0, L_1) \cup (L_1 + L_2, 1 - e) \cup (1 - e, 1 - L_5)$, and reserves orientation in $(L_1, L_1 + L_2) \cup (1 - L_5, 1)$, here $F = \{2, 5\}$ and $\tau(1) = 3$, $\tau(2) = 4$, $\tau(3) = 5$, $\tau(4) = 1$, and $\tau(5) = 2$.

Let A be the matrix given by

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix}. \tag{1.2}$$

The characteristic polynomial, $p(\lambda)$, of the matrix A is

$$p(\lambda) = \det(\lambda I - A) = (\lambda^2 - 1)\, q(\lambda),$$

where $q(\lambda)=\lambda^3-3\lambda^2+\lambda-1$. Since $q(1)=-2$, $q(\lambda)$ has a real root $\lambda>1$. With respect to this eigenvalue λ, we note that A has eigenvectors l with positive coordinates, we take l such that

$$L=Al=\lambda l \tag{1.3}$$

is a probability vector.

Now we define T according to (1.1) using L (1.3), and we have the following result:

PROPOSITION 1. *The forward Poincaré map induced by T on the interval*

$$(L_1, L_1+e) \tag{1.4}$$

is equal to the transformation T linearly contracted to this interval.

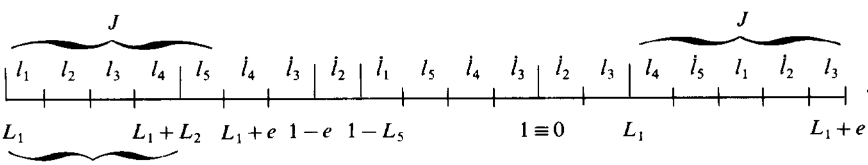
Proof. We have

$$L=(l_2+l_3, l_1+l_2+l_3+l_4, l_3+l_4+l_5, l_2, l_1+l_3+l_4+l_5)$$

and note that, by (1.1),

$$e=l_1+l_2+l_3+l_4+l_5=\frac{1}{\lambda}.$$

We now obtain the forward Poincaré map T_1, induced by T, on the interval $J=(L_1, L_1+e)$. We begin from the second interval



(1.5)

The above partition gives the segments which are invariant over the first return map T_1, the l_i indicate their lengths and the point above the segment, $\dot{l}_i$, means that its orientation has been reversed. Thus we have the

the forward Poincaré map T_1 acts on the interval J as a contraction of T by e. ∎

An analogous way to write (1.5) is

$$\overbrace{1\ 2\ 3\ 4}^{J}\ \ 5\ \dot{4}\ \dot{3}\ \ \dot{2}\ \ \dot{1}\ 5\ \dot{4}\ \dot{3}\ \Big|_{1\equiv 0}\ \dot{2}\ 3\ \overbrace{4\ \dot{5}\ 1\ \dot{2}\ 5}^{J}. \tag{1.6}$$

orientation
reversed
intervals

Here the numbers 1, 2, 3, 4, and 5 represent, respectively, segments of length l_1, l_2, l_3, l_4, and l_5, and the point above the number means orientation reversed.

By (1.3), l is an eigenvector of A,

$$l = A\frac{1}{\lambda}l,$$

therefore we can apply Proposition 1 to the forward Poincaré map T_1, and so on. Once we have done this, we obtain a decreasing sequence of intervals $J_1 = J \supset J_2 \supset \cdots \supset J_n \supset \cdots$, such that $|J_{n+1}| = e\,|J_n|$, where $|J|$ means the length of J. Moreover, the forward Poincaré map induced by T on J_n is equal to T contracted by e^n. Therefore, $\bigcap_{n \geqslant 1} J_n$ is a one-point set, say $\{p\}$. Note that $Tx \neq x$, for all $x \in D$, so the positive orbit of p, $\mathcal{O}_+(p) = \{T^n p : n = 1, 2, \ldots\}$ is infinite, moreover $p \in \overline{\mathcal{O}_+(p)}$.

We can extend the above construction (1.6) to obtain exchanges of more than 5 intervals with the same properties as T. Here we omit the inductive argument which permits us to define such interval exchanges.

For $n \geqslant 2$, let τ be the permutation on $\{1, \ldots, 2n+1\}$ given by

$$\tau(i) = \begin{cases} i+2, & \text{for } \ 1 \leqslant i \leqslant 2n-1 \\ i-2n+1 & \text{otherwise} \end{cases}$$

and $F = \{2, 4, \ldots, 2n-2, 2n+1\}$ be the flip set. Using τ and F, we define exchanges of $2n+1$ intervals which flip n intervals, which flip n intervals, namely T. According to (1.6), we begin from the second interval and our goal is to split J into $2n+1$ intervals

$$\underbrace{\overbrace{1\ 2\ 3\ 4\ 5\ 6\cdots(2\dot{n}-1)(2n)}^{J}}_{\text{2nd interval}}\ \underline{(2n+1)(2\dot{n})\cdots\dot{6}\ \dot{5}\ \dot{4}\ \dot{3}}\ \underbrace{\dot{2}}\ \underline{\dot{1}(2n+1)(2\dot{n})\cdots\dot{6}\ \dot{5}}\ \underbrace{\dot{4}}\ \underline{\dot{3}\ \dot{2}\ \dot{1}(2n+1)\cdots\dot{8}\ \dot{7}}\ \underbrace{\dot{6}}\cdots$$

oriented
reversed
intervals

$$\cdots\underbrace{(2n-2)}_{(2n)\text{th interval}}\ \underbrace{(2\dot{n}-3)(2n-4)(2\dot{n}-5)\cdots4\ \dot{3}\ 2\ \dot{1}(2n+1)(2\dot{n})(2\dot{n}-1)}_{(2n+1)\text{st interval}}\ \Big|_{1\equiv 0}\ \underbrace{(2\dot{n}-2)(2n-1)}_{\text{1st interval}}\ \overbrace{(2n)\ (2\dot{n}+1)\ 1\ \dot{2}\ 3\ \dot{4}\ 5\cdots(2\dot{n}-4))(2n-3)(2\dot{n}-2)(2n-1)}^{J} \tag{1.7}$$

orientation
reversed
interval

The notation used here is the same as in (1.6).

Therefore the first return Poincaré map induced on J by T behaves as T. We call $I_1, ..., I_{2n+1}$ the open intervals exchanged by T, $I_1, ..., I_{2n+1}$ form a partition of $[0, 1)$, and $J_1, ..., J_{2n+1}$ the smaller open intervals which partitioned J, for $1 \leqslant i, l \leqslant 2n+1$ let

$$A_{i,l} = \# \text{ of times the interval } J_l \text{ appears at } I_i \text{ in (1.7)}, \tag{1.8}$$

and A be the $(2n+1) \times (2n+1)$ matrix whose entries are A_{il}. Let $L_i = |I_i|$ and $l_i = |J_i|$, for $1 \leqslant i \leqslant 2n+1$, and $L = (L_1, ..., L_{2n+1})$ and $l = (l_1, ..., l_{2n+1})$, we have

$$L = Al.$$

We note that the following entries of A (1.8) are equal to 1

$$A_{1,2n-2}, A_{2n-2,2n-4}, ..., A_{4,2}, A_{2,3}, A_{3,5}, ...,$$
$$A_{2n-1,2n+1}, A_{2n+1,1}, A_{2n+1,2n}, A_{2n,2n-2}.$$

According to [12], this means that the matrix A is irreducible. By (Proposition 2, p. 535, [12]), since the matrix A is nonnegative and irreducible, it has an eigenvalue $\lambda_0 \geqslant r$, where r can be chosen in the following manner. Take $\varepsilon = 2^{1/n} - 1$ and $x = (x_1, ..., x_{2n+1})$ such that

$$x_1 = x_3 = \cdots = x_{2n+1} = 1$$
$$x_{2i} = (1+\varepsilon)^{n-i} \qquad \text{for} \quad 1 \leqslant i \leqslant n.$$

Let

$$r = r(x) = \min_{1 \leqslant i \leqslant 2n+1} \frac{(Ax)_i}{x_i},$$

therefore $r \geqslant 1 + \varepsilon > 1$, since

$$\frac{(Ax)_i}{x_i} \geqslant \begin{cases} 2 & \text{iff odd} \\ 1+\varepsilon & \text{otherwise.} \end{cases}$$

Besides there exists $y = (y_1, ..., y_{2n+1}) \in \mathbb{R}^{2n+1}$, with $y_i > 0$, such that $\lambda_0 = r(y)$ and $Ay = \lambda_0 y$.

Remark 1. For simplicity, from now on we deal only with the case $n = 2$, although the results which will be stated hold for any case with $n \geqslant 2$.

According to [5], we can relate the transformation T (Proposition 1) to a C^∞ vector field on the torus with two cross-caps.

Let $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ and $C = \mathbb{Z} \times \mathbb{R}/\mathbb{Z}^2$. Let D_1 and D_2 be disjoint open disks in $T^2 - C$ with boundary circles S_1 and S_2. Let f_1 and f_2 be the antipodal maps of S_1 and S_2. Let $\tilde{S}_1 = S_1/f_1$ and $\tilde{S}_2 = S_2/f_2$ be the quotients of S_1 and S_2 by f_1 and f_2, respectively. The surface

$$M^2 = (T^2 - D_1 \cup D_2)/f_1 \cup f_2 \tag{1.9}$$

is a torus with two cross-caps.

So for each transformation T, given by (1.1), we can construct a C^∞ vector field X on M with the following properties:

(i) X has only two singularities and both are saddle points.

(ii) C is a circle transverse to X.

(iii) Any point of $\{L_1, L_1 + L_2, 1 - e, 1 - L_5\}$ goes straight to a saddle point and does not intersect the interval $(0, 1) \subset C$.

(iv) The forward Poincaré map induced by X on C is given by T.

If we take T which satisfies Proposition 1, the following result (see [5]) hols for X:

THEOREM A. *Any positive (negative) semitrajectory of X, with the exception of the saddle points and the points which are in the stable (unstable) separatrices of the saddle points, is nonorientable dense.*

Using the method developed here we can obtain denumerable many interval exchange transformations like T for which Proposition 1 holds.

For a fixed $n = 0, 1, 2, \ldots$, let A_n be the matrix

$$A_n = \begin{pmatrix} n & n+1 & n+1 & n & n \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix}. \tag{1.10}$$

Note that for $n = 0$, we obtain (1.2). For simplicity, we write A in place of A_n. The characteristic polynomial of A, $p(\lambda)$, is

$$p(\lambda) = (\lambda^2 - 1)(\lambda^3 - (3 - n)\lambda^2 + (1 - n)\lambda - 1),$$

therefore $p(\lambda)$ has a real root $\lambda > 1$.

Let l be the eigenvector of A associated to λ such that

$$L = Al = \lambda l$$

is a probability vector. We use L to define, according to (1.1), the transformation T. Therefore, Theorem A also holds for the C^∞ vector field X which is constructed from T.

2. Nondenumerable Many Nonorientable Flows

We use expression (1.6) and remark therefore in classifying two types of transformation (1.1),

$$\text{Type 0:}\quad \underbrace{\overbrace{1\ 2\ 3\ 4}^{J}}\ \underline{5\ \dot{4}\ \dot{3}}\ \underline{\dot{2}}\ \underbrace{\dot{1}\ 5\ \dot{4}\ \dot{3}}\ \Big|_{1\equiv 0}\ \underline{\dot{2}\ 3}\ \overbrace{4\ \dot{5}\ 1\ \dot{2}\ 3}^{J} \tag{2.1a}$$

$$\text{Type 1:}\quad \underbrace{\overbrace{1\ 2\ 3\ 4}^{J}}\ \underline{5\ \dot{4}\ \dot{3}}\ \underline{\dot{2}}\ \underbrace{\dot{1}\ 5\ \dot{4}\ \dot{3}}\ \Big|_{1\equiv 0}\ \underline{\dot{2}\ 3\ 4\ \dot{5}\ 1\ \dot{2}\ 3}\ \overbrace{4\ \dot{5}\ 1\ \dot{2}\ 3}^{J}. \tag{2.1b}$$

We say that T is of Type 0 or Type 1, if T can be described by the expression (2.1a) or (2.1b), respectively.

For $n=0$ or 1 fixed, let A_n be the matrix given by (1.10). Let $l=(l_1, l_2, l_3, l_4, l_5)$ be a vector such that

$$L=(L_1, L_2, L_3, L_4, L_5)=A_n l \tag{2.2}$$

is a probability vector. Using L, we define by (1.1) the transformation $T_0=T$. Therefore, according to the choice of n in (2.2), T_0 is of Type 0 or Type 1.

Let T_1 be the forward Poincaré map, induced by T_0, on the interval $I_1=(L_1, L_1+e)$. Recall that $e=L_4+L_5=l_1+l_2+l_3+l_4+l_5$. We can identify T_1 with an interval exchange transformation on the interval (0, 1) defined by

$$x \mapsto \frac{1}{e}\, T_1(L_1+ex).$$

In this definition, for simplicity we have omitted the fact that the transformation is not defined in the four endpoints of the exchanged intervals. Note that they are also omitted in the definition (1.1).

So we can take T_1 as been defined by (1.1), for $L=l/\|l\|_1$. Analogously to T_0, we can take T_1 to be of Type 0 or Type 1. In this way we can defined by induction an infinite sequence of transformations

$$T_0, T_1, T_2, ..., T_n, T_{n+1}, ..., \tag{2.3}$$

where T_{n+1} is of Type 0 or Type 1 and it is defined from T_n, for $n = 1, 2, \ldots$, as T_1 was defined from T_0. The sequence (2.3) uniquely defines a transformation T_0, so we have the following result:

PROPOSITION 2. *To each $x \in \{0, 1\}^{\mathbb{Z}_+}$, we can associate a transformation $T = T_0$ given by* (1.1) *such that the* $(n+1)$th *member, T_n, of the sequence* (2.3) *is of Type $x(n)$, $n = 0, 1, 2, \ldots$.*

Due to Proposition 2 and Remark 1, we obtain the following extension of Theorem A.

THEOREM 2. *On any nonorientable surface of genus $n \geqslant 4$, there are nondenumerable many smooth vector fields with nonorientable nontrivial recurrent trajectories which do not give rise—through "cut and paste"—to vector fields on a surface of genus smaller than n.*

Proof. Here we consider a surface of genus 4, for any other surface the proof works the same.

Let M^2 be the torus with two cross-caps (1.9) and C the circle in M^2 defined in Section 1. To each transformation T obtained through Proposition 2 we associate a C^∞ vector field X on M^2 (see Sect. 1) such that T is the forward Poincaré map induced by X on C. Later, we will show that X satisfies Theorem A. We note that there are only denumerable many circles in M^2 conjugated to C. So up to conjugation, we still have nondenumerable many distinct C^∞ vector fields which are suspensions of the transformations which satisfy Proposition 2.

In order to conclude Theorem 2, we just have to show that each C^∞ vector field X has nonorientable nontrivial recurrent trajectories. We prove here the recurrence.

Let T_0 be one of the transformations constructed to prove Theorem 2. Let $(L_1, L_2, L_3, L_4, L_5)$ be the probability vector associated to T_0. As we have already seen, T_0 induces a forward Poincaré map T_1 in the interval $J_1 = (L_1, L_1 + e)$. T_1 is associated to the vector $l = (l_1, l_2, l_3, l_4, l_5) = A_n^{-1} L$ (2.2), for $n = 0$ or 1, whatever is the type of T_0. According to (2.1), we have

$$1 = \begin{cases} 2l_1 + 3l_2 + 4l_3 + 3l_4 + 2l_5, & \text{if } T_0 \text{ is of Type 0,} \\ 3l_1 + 4l_2 + 5l_3 + 4l_4 + 3l_5, & \text{if } T_0 \text{ is of Type 1,} \end{cases}$$

therefore $|J_1| = e = l_1 + l_2 + l_3 + l_4 + l_5 < \frac{1}{2}$.

The transformation T_{n+1}, the $(n+2)$th term of the sequence (2.3), can be identified with the forward Poincaré map, induced by the copy of T_n, on the interval J_{n+1}, therefore $|J_{n+1}| < 1/2^{n+1}$. Furthermore, $J_1 \supset J_2 \supset \cdots \supset J_n \supset \cdots$ is a decreasing sequence of intervals and $\bigcap_{n \geqslant 1} J_n$ is a one-point set, say $\{p\}$. Since, for any T_0 defined by (1.1), $Tx \neq x$, for all $x \in D$ (1.1), and $|J_n| \to 0$, as $n \to \infty$, we have that the positive orbit of P,

$$\mathcal{O}_+(p) = \{T_0^k p: k = 1, 2, ...\}$$

is infinite and $p \in \overline{\mathcal{O}_+(p)}$. Note that any forward Poincaré map of T_0 is defined by positive powers of T_0 restricted to certain intervals.

This proves the nontrivial recurrence.

Following the proof of Theorem A given in [5], we can see that X satisfies all other properties.

3. Generic Property of Recurrent Flows on M^2

In [11], Gutierrez studies the class of flows that are, up to topological equivalence, smooth suspensions of interval exchange transformations with a dense semi-orbit. He called such object a basic Cherry flow: a continuous flow defined on a compact connected two-manifold such that one of the following properties holds:

(i) it has finitely many fixed points (singularities) and a dense positive semi-orbit,

(ii) it has been obtained by blowing up finitely many dense positive semi-trajectories of a flow which satisfies (i).

Let $\mathcal{V}$ be the set of all C^∞-vector fields on the nonorientable surface M^2 (1.9) which induces on the circle $C \subset M^2$ Poincaré maps which are suspensions of interval exchange transformations. We will prove that the so-called basic Cherry flows form a vanishing set in $\mathcal{V}$, with respect to the Lebesgue measure, on the set

$$\{\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5): \alpha \text{ is a probability vector}\}.$$

Our result also holds for any nonorientable surface of genus $\geqslant 4$.

Theorem 3. *The vector fields in $\mathcal{V}$ with a trajectory which is a nonorientable closed orbit, form an open dense set in $\mathcal{V}$ in the C^∞-topology. Moreover, it is a full measure set in $\mathcal{V}$.*

In order to prove Theorem 3, we need two results. The first is due to the author [10].

Theorem B. *Let $n > 1$ and τ be permutation on $\{1, 2, ..., n\}$. Let*

$$\Omega = \{\alpha \in \mathbb{R}^n: \alpha \text{ is a probability vector}\}.$$

Let F be a nonempty flip set and $\mathcal{T}(\sigma, F)$ be the space of all interval exchange transformations defined by $\alpha \in \Omega$ and τ with flip set F. Then, with respect to the Lebesgue measure on Ω, almost all transformations in $\mathcal{T}(\sigma, F)$ are periodic on a nonempty open interval.

The second result is a trivial remark on the nonorientability of the surface M^2 for which we omit the proof, and it holds for any nonorientable surface.

PROPOSITION 3. *Let X be a C^∞ vector field in $\mathscr{V}$ such that any trajectory of X is a closed orbit, a saddle connection, or a saddle point, then X has at least one nonorientable closed orbit, or one nonorientable saddle connection.*

We now prove Theorem 3.

Let $X \in \mathscr{V}$ and T be the interval exchange transformation defined on the set D, the interval $(0, 1)$, but finitely many singular points, which is associated to X. Moreover, we assume that any trajectory of X is a closed orbit, a saddle connection, or a saddle point. By Proposition 3, at least one of the closed orbits is a nonorientable trajectory, or there exists a nonorientable saddle connection. If the last occurs, in terms of T this means that there exists $x \in D$, $k = 1, 2, \ldots$, and $\varepsilon > 0$ such that

$$T^k x = x \qquad (x \text{ is a fixed point for } T^k), \tag{3.1a}$$

$$T^k(x-\varepsilon, x+\varepsilon) = (x-\varepsilon, x+\varepsilon) \text{ and } T^k \text{ reverses orientation in } (x-\varepsilon, x+\varepsilon)\ (T^k y = 2x - y \text{ for all } x-\varepsilon < y < x+\varepsilon). \tag{3.1b}$$

We first note that T^k is itself an interval exchange transformation, so for simplicity we can take $k = 1$.

Let T be an interval exchange transformation which is defined in Section 1, with the probability vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and the irreducible permutation τ, that is, $\tau = \{1, \ldots, i\} \neq \{1, \ldots, i\}$, for all $i < n$. We assume that for a fixed $1 \leqslant i \leqslant n$, $I_i \cap T(I_i) \neq \varnothing$, and T reverses orientation in I_i, here (β_{i-1}, β_i) is an open interval. Therefore T has a fixed point in I_i and satisfies (3.1).

We take

$$0 < \varepsilon < \tfrac{1}{4} \min\{|I_i \cap T(I_i)|, \min_{j \neq i} \alpha_j\}. \tag{3.2}$$

Let $(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{n} a_j = 0 \qquad \text{and} \qquad \sum_{j=1}^{n} |a_j| < \varepsilon.$$

We now define, using the probability vector $(\alpha_1 + a_1, \alpha_2 + a_2, ..., \alpha_n + a_n)$ and the permutation τ, the interval exchange transformation T' which only reverses orientation where T does.

Let I_i' be the ith open interval associated to the transformation T'. Therefore T' reverses orientation in I_i'. Furthermore we have by our choice of ε (3.2) that

$$|I_i' \cap T'(I_i')| > |I_i \cap T(I_i)| - 2\varepsilon > 0.$$

This proves that T' satisfies (3.1). Let X' be the C^∞-vector field on M^2 which is the suspension of T', so any trajectory of X' is a closed orbit, a saddle connection or a saddle point. Therefore these vector fields form an open set in $\mathscr{V}$, in the C^∞ topology.

Let X induce a transformation T such that the intervals exchanged by T have all rational lengths, so any trajectory of X is a closed orbit, a saddle connection or a saddle point. Furthermore these vector fields form a dense set in $\mathscr{V}$ in the C^∞ topology.

Let X induce a transformation T that has a saddle connection, therefore the lengths of the five intervals exchanged by T are linearly dependent with respect to Q, the rational numbers. So such vector fields X form a null set, with respect to the Lebesgue measure on Ω. Therefore, by Theorem B, the vector fields X in $\mathscr{V}$ such that any trajectory of X is a closed orbit or a saddle point, form a full measure set. Furthermore the ones which have a nonorientable closed orbit form an open set with full Lebesgue measure, thus this set is also dense.

This proves the theorem. ∎

Acknowledgments

The author is indebted to J. Palis for valuable suggestions and many helpful discussions on the subject of this paper. The author would like also to thank J. C. Yoccoz, A. Lins Neto, and M. J. D. Carneiro for useful discussions during the preparation of this paper.

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